

# Volumes of Solids of Revolution. A Unified Approach

Jorge Martín-Morales\* and Antonio M. Oller-Marcén  
jorge@unizar.es, oller@unizar.es

Centro Universitario de la Defensa - IUMA.  
Academia General Militar, Ctra. de Huesca s/n.  
50090, Zaragoza, Spain.

## 1 Introduction

The computation of the volume of solids of revolution is a very common topic in undergraduate calculus courses [4, 6]. Usually two methods are presented in textbooks, namely:

1. The disk method, which roughly consists of decomposing the solid into slices that are perpendicular to the axis of revolution.
2. The shell method, that considers the solid as a series of concentric cylindrical shells wrapping the axis.

From a geometrical point of view these two methods are quite different and it is the shape of the solid what motivates the choice between one or another. Nevertheless, from an analytical point of view, both methods are equivalent and they can be related using the well-known formula for integration by parts [1], inverse functions [3] or even Rolle's theorem [2].

We wondered if there was an even deeper relation between the methods above and we have found that this is indeed the case. In particular we present here a method to compute the volume of a solid of revolution as a double integral in a very simple way. Then, we see that the classical methods (disks and shells) are recovered if this double integral is computed by each of the two possible applications of Fubini's theorem. As a further application we also show how Pappus' theorem is obtained from our formula.

---

\*Partially supported by the projects MTM2010-21740-C02-02, "E15 Grupo Consolidado Geometría" from the government of Aragón, FQM-333 from "Junta de Andalucía", and PRI-AIBDE-2011-0986 Acción Integrada hispano-alemana.

## 2 The volume as a double integral

Let  $S$  be a closed region of the plane  $OXY$  and let  $e$  be any straight line in the same plane such that  $e$  is exterior to  $S$ . For every point  $P = (x, y) \in S$ , put  $d_e(x, y) = d(P, e)$  the distance from  $P$  to  $e$ . Let us denote by  $V(S, e)$  the volume of the solid obtained by rotating the region  $S$  around the line  $e$ , see Figure 1.

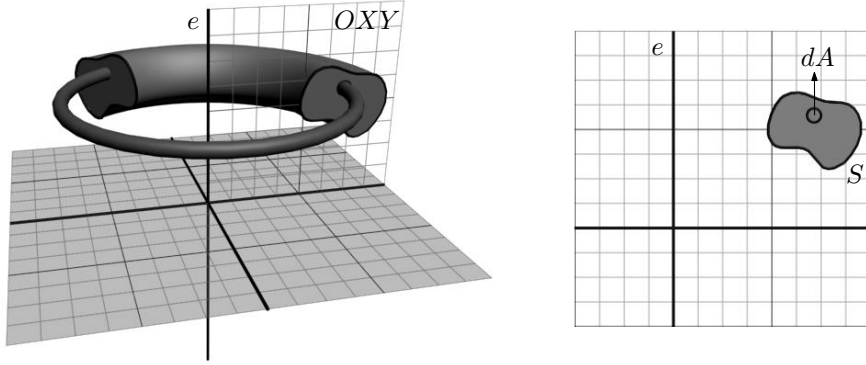


Figure 1: Solid of revolution together with the rotating section.

We claim that:

$$V(S, e) = \iint_S 2\pi d_e(x, y) dA. \quad (1)$$

It is not the point here to give a complete and rigorous proof of this claim. The underlying idea is in fact very simple. For every point  $P(x, y) \in S$ , consider a tiny circle with center in  $P$  and with area  $dA$  (see the right hand side of Figure 1). When this circle rotates around the axis  $e$  it generates a torus of volume  $2\pi d_e(x, y)dA$  and then it is enough to sum up all these volumes; i.e., to integrate over  $S$ .

Observe that  $d_e(x, y)$  is always a polynomial of degree 1 in  $x$  and  $y$ , namely, if the axis  $e$  has equation  $ax + by + c = 0$ , then  $d_e(x, y) = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}$ .

## 3 Deducing the “standard” methods

In this section it is shown how the classical disk and shell methods can be obtained from the double integral formula presented above.

Assume, without loss of generality, that the axis of revolution is  $OY$ . Let us first consider  $S$  a normal domain with respect to the  $x$ -axis, i.e., the region  $S$  is bounded by continuous functions  $y = f_1(x)$  and  $y = f_2(x)$  between  $x = a$  and  $x = b$  as in the left-hand side of Figure 2. Fubini’s

theorem states that the double integral  $I = \iint_S 2\pi x \, dA$  can be computed by means of simple integrals as:

$$I = \int_a^b \left( \int_{f_1(x)}^{f_2(x)} 2\pi x \, dy \right) dx = \int_a^b 2\pi (f_2(x) - f_1(x)) \, dx,$$

which is precisely the well-known formula obtained in the method of shells.

Assume now that  $S$  is a normal domain with respect to the  $y$ -axis. In this case  $S$  is bounded by continuous functions  $x = g_1(y)$  and  $x = g_2(y)$  between  $y = c$  and  $y = d$  as in the right-hand side of Figure 2. Again, by Fubini, the double integral  $I$  above can be solved as:

$$\iint_S 2\pi x \, dA = \int_c^d \left( \int_{g_1(y)}^{g_2(y)} 2\pi x \, dx \right) dy = \int_c^d \pi (g_2(y)^2 - g_1(y)^2) \, dy,$$

which is the formula corresponding to the disk method.

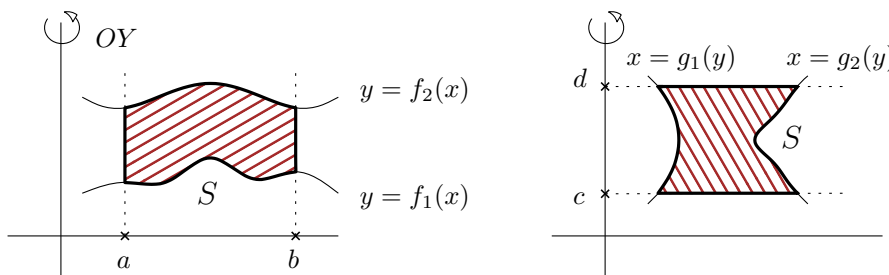


Figure 2: Normal domains w.r.t. the  $x$ - and the  $y$ -axis, resp.

This shows that, although both methods might seem very different geometrically (slicing the solid versus decomposing it in cylindrical shells), they are essentially the same. This is not surprising and has a simple interpretation. Using the formula  $\iint_S 2\pi x \, dA$  one is adding up the volumes of the tori obtained by rotating a circle with center  $(x, y)$  and area  $dA$  about the  $y$ -axis. To sum all these volumes one can proceed in two different ways:

1. Fix  $y = y_0$  and consider the sum of the volumes of the tori corresponding to the points  $(x, y_0)$ , obtaining a horizontal disk. Then it is enough to sum the volume of these disks. This is exactly the description of the disk method.
2. Fix  $x = x_0$  and consider the sum of the volumes of the tori corresponding to the points  $(x_0, y)$ , obtaining a vertical cylindrical shell. Then one has just to sum the volumes of this shells. This is exactly the description of the shell method.

Hence the two classical methods arise from ours just by carefully arranging the tori whose volumes have to be summed.

## 4 Pappus theorem

Consider a region  $S$  of the plane  $OXY$  and the solid obtained by rotating it around the axis  $e$ . If  $C$  denotes the centroid of  $S$  and  $\mathcal{A}$  is the surface area of  $S$  (recall the notation from Section 2), then the so-called Pappus' theorem states in its classical form [5, Chapter 6] that the volume of this solid is given by

$$V(S, e) = 2\pi d(C, e)\mathcal{A}.$$

Let us see how to obtain Pappus' theorem from Equation (1). Recall that if  $e$  has equation  $ax + by + c = 0$ , then  $d_e(x_0, y_0) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$ . It is also well-known that if  $C = (x_C, y_C)$ , then:

$$x_C = \frac{\iint_S x \, dA}{\mathcal{A}}, \quad y_C = \frac{\iint_S y \, dA}{\mathcal{A}}.$$

With all this ingredients, and since  $S$  can be assumed to lie on the semiplane determined by  $ax + by + c > 0$ :

$$\begin{aligned} V(S, e) &= \iint_S 2\pi d_e(x, y) \, dA = 2\pi \iint_S \frac{ax + by + c}{\sqrt{a^2 + b^2}} \, dA = \\ &= 2\pi \mathcal{A} \frac{ax_C + by_C + c}{\sqrt{a^2 + b^2}} = 2\pi d(C, e)\mathcal{A}, \end{aligned}$$

where linearity of the integral has played a key role.

Pappus' theorem admits an interesting geometrical interpretation. When the region  $S$  is rotated around  $e$ , a torus with section  $S$  is obtained. Then Pappus' theorem implies that the volume of this torus is the same as that of a cylinder with base  $S$  and height  $2\pi d(C, e)$ . In our case we have a similar interpretation, but the volume of the original torus is now the same as that of a truncated cylinder. Figure 3 illustrates both situations.

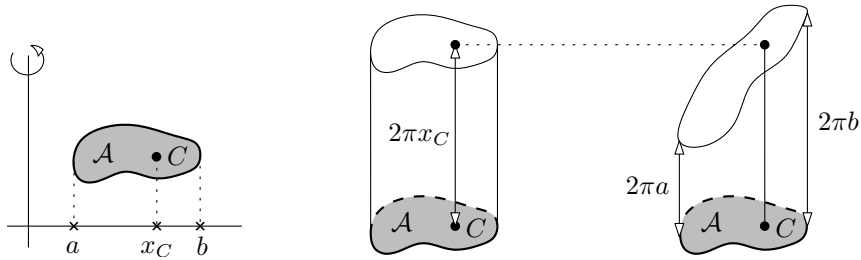


Figure 3: Pappus' theorem & our method.

## 5 An example

As shown in Section 3, the formula for computing the volume of a solid of revolution using a double integral gives rise to the disk or shell method

after applying Fubini's theorem. This fact could lead us to think that this approach does not provide in principle any new advantage with respect to the classical ones.

However, in order to solve a double integral, there are many more techniques at our disposal. The possibility to use specific double integration techniques, as for instance change of variables, is what gives merit to our method. An illustrative example based on changing to polar coordinates is shown.

Let  $S$  be the plane region bounded by the unit circle  $x^2 + y^2 = 1$  and the straight lines  $y = x$  and  $y = -\sqrt{3}x$  as in Figure 4. Let us compute, using integrals, the volume of the solid obtained by rotating the region  $S$  about the axis  $OY$ .

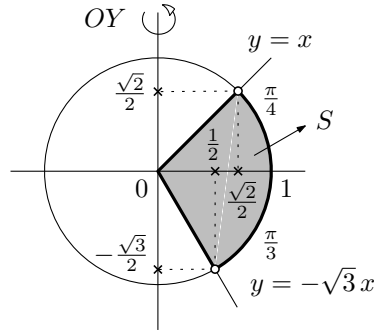


Figure 4: Area of the rotating region about the  $y$ -axis.

- Using the disk method, the required volume is:

$$V = \int_{-\frac{\sqrt{3}}{2}}^0 \pi \left( 1 - y^2 - \frac{y^2}{3} \right) dy + \int_0^{\frac{\sqrt{2}}{2}} \pi \left( 1 - y^2 - y^2 \right) dy. \quad (2)$$

- Using the method of shells, the volume is:

$$\begin{aligned} V = & \int_0^{\frac{1}{2}} 2\pi x \left( x + \sqrt{3}x \right) dx + \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} 2\pi x \left( x - \sqrt{1 - x^2} \right) dx \\ & + \int_{\frac{\sqrt{2}}{2}}^1 2\pi \left( 2\sqrt{1 - x^2} \right) dx. \end{aligned} \quad (3)$$

- Finally, using our method, the volume is:

$$V = \iint_S 2\pi x \, dA. \quad (4)$$

Observe that the last expression using double integrals is much simpler and more compact. Applying Fubini's theorem one recovers the two other expressions and there would be no real advantage in our method. Instead, let us change to polar coordinates.

Then,

$$V = 2\pi \int_0^1 \rho^2 d\rho \int_{-\frac{\pi}{3}}^{\frac{\pi}{4}} \cos \theta d\theta = 2\pi \left[ \frac{\rho^3}{3} \right]_0^1 \left[ \sin \theta \right]_{-\frac{\pi}{3}}^{\frac{\pi}{4}} = \frac{\pi(1 + \sqrt{2})}{3},$$

and thus the required volume can be found in a simple way compared to the expressions (2) and (3).

## 6 Concluding remarks

We have presented a way to compute the volume of a solid of revolution as a double integral. This method seems, as far as we know, to be absent in the literature. In addition to its intrinsic mathematical interest, we think that this method to compute the volume of a solid of revolution might be of educational interest. Specifically, some of its advantages are:

1. It avoids considerations about the shape of the solid.
2. It gives an easy way to describe the volume of the solid when the axis of revolution is not horizontal neither vertical.
3. It introduces the use of double integration techniques (polar coordinates, for instance) that, in some cases (see the example in Section 5 above), leads to shorter calculations.
4. It extends the disk and shell methods and also Pappus' theorem in the sense that all of them can easily be recovered from Equation (1).

**Acknowledgments.** We would like to thank Rubén Vígara for his help with Figure 1.

## References

- [1] Charles A. Cable. The Teaching of Mathematics: The Disk and Shell Method. *Amer. Math. Monthly*, 91(2):139, 1984.
- [2] Walter Carlip. Disks and shells revisited. *Amer. Math. Monthly*, 98(2):154–156, 1991.
- [3] Eric Key. Disks, shells and integrals of inverse functions. *College Math. J.*, 25(2):136–138, 1994.

- [4] Ron Larson and Bruce H. Edwards. *Calculus*. Brooks/Cole, Belmont, ninth edition, 2010.
- [5] Salas, Saturnino L., Hille Einar and Garret J. Etgen. *Calculus: One and Several Variables*. Wiley & Sons, Toronto, eighth edition, 1998.
- [6] James Stewart. *Calculus: Concepts & Contexts*. Brooks/Cole, Belmont, fourth edition, 2010.